Chapter 8

Managing the Uneven-Aged Forest with Linear Programming

FROM: BUONGIORNO AND GILLESS
FOREST MANAGEMENT AND ECONOMICS

8.1 INTRODUCTION

In an *uneven-aged* or selection forest, many trees of different age and size coexist on small tracts of land. In contrast with an even-aged forest, distinct areas of homogeneous age classes cannot be distinguished. The ideal uneven-aged forest, where trees of all ages appear on the same acre, is however, rare. Often, trees may be grouped in patches of similar age, but the patches are too small to be managed like the even-aged compartments that we used in the previous chapters.

Large tracts are never clear-cut in the uneven-aged forest. Rather, one selects within stands single trees or group of trees. Consequently, unlike even-aged stands, uneven-aged stands have no beginning and no end. There are always trees on each acre of the uneven-aged forest, even immediately after harvest.

In a selection forest, regeneration is mostly natural. It comes from the stock of saplings in the understory emerging through the openings left by cutting the large trees. Therefore, this form of management works best with trees that are shade tolerant, for example, maples, hemlocks, cedars, spruces, and firs. Nevertheless, many forests of ponderosa pine in western United States are uneven-aged, despite the fact that the species needs light for good regeneration. In that case, instead of the pure form of selection cutting, trees are cut in small patches, leading to an overall structure that is essentially uneven-aged for management purposes.

Uneven-aged management leads to a forest with a more natural aspect than its even-aged counterpart. For that reason, it is very attractive for forests managed for multiple use, including recreation. For small, private woodlots, it is often the only form of cutting that is acceptable. In that case, a good-looking forest not only pleases the owner, but it often enhances the value of his property.

Unfortunately, uneven-aged management is usually believed to be inferior from a purely economic point of view. Of course, this is certainly not the case for a woodlot in which timber production is only a secondary object. But even for pure timber management, the case against uneven-aged management is not that clear. First of all, for some species, it is the only possible silviculture, if any regeneration is to be obtained at all. The starting of a new crop of good trees is the most costly operation in forestry. In uneven-aged management this cost is minimal.

On the other hand, the costs of harvesting, per unit of volume, are indeed generally greater in a selection forest. There are two reasons for this. First, more area must be covered to extract a given volume than by clear-cutting. This means higher costs for roading and movement of machinery and people. Second, the felling of trees and hauling of logs is a more delicate operation in a selection forest. Care must be taken not to damage the trees that are left, especially young saplings that will constitute the future crops. This is a labor-intensive process that can be done only by skilled workers and that is difficult to mechanize. For this reason, uneven-aged management is most appropriate for the production of large trees leading to expensive, high-quality timbers for which the cost of harvesting represents a small part of the value of the final product.

Perhaps because they are more complex than even-aged systems, selection cutting systems have not been studied as much. Relatively few models of selection forests exist, and little is known about the real economics of these forests for timber production. The object of this chapter is to study such a model, originally developed by Buongiorno and Michie (1980), and to use it to investigate problems of interest to forest managers. These problems include the length of the cutting cycle, i.e., the interval between successive cuttings on a given tract of land, and the intensity of the cut, i.e., the number and size of trees to be removed.

8.2 A GROWTH MODEL OF THE UNEVEN-AGED FOREST STAND

The model deals with an uneven-aged stand. A *stand* is an area small enough to be cut within a short period of time, say a year. Thus, a stand could be the entire woodlot of a farmer, or one-twentieth of a large industrial forest managed on a 20-year cutting cycle. The state of a stand is described by the diameter distribution of trees on the average acre. Usually, this distribution is determined from a few sample plots.

Table 8.1 shows the diameter distribution of a managed sugar-maple stand in the Lake States. To lighten notations, only three diameter classes have been used. In practice, six or seven classes are often necessary, but the

TABLE 8.1 DIAMETER DISTRIBUTION OF AN UNEVEN-AGED SUGAR MAPLE STAND

Diameter class, i	Diameter range (inches)	Number of trees	Average diameter (inches)	Basal area of tree (ft^2)	Total basal area (ft^2 / a)
1	4.0–7.9	420	6.0	0.20	84
2	8.0–13.9	117	11.0	0.66	77
3	14.0 +	7	16.0	1.40	10
Total		544			171

principles remain the same. The table shows the typical inverse J shape of the diameter distribution in an uneven-aged stand, with many small trees and a few large ones.

In this model, the state of the stand at any point in time is represented by three variables: $y_{1,t}$, $y_{2,t}$, $y_{3,t}$, where $y_{i,t}$ is the number of live trees per acre in diameter class i at time t. A growth model is then a set of equations that gives the state of the stand at time $t + 1$, given its current state. Our model consists of three equations:

$$
\begin{aligned}
y_{1,t+1} &= a_1 y_{1,t} + O_t \\
y_{2,t+1} &= b_1 y_{1,t} + a_2 y_{2,t} \\
y_{3,t+1} &= \qquad\qquad b_2 y_{2,t} + a_3 y_{3,t}
\end{aligned}
\tag{8.1}
$$

where the variable O_t in the first equation is the *ingrowth*, the number of young trees that enter the first diameter class during the interval t to $t + 1$.

Each parameter a_i is the fraction of live trees in diameter class i at t that are still alive and in the same diameter class at $t + 1$. Each parameter b_i is the fraction of live trees in diameter class i at t that are alive, but in diameter class $i + 1$ at $t + 1$.

Consequently, the fraction of trees in age class i at t that are dead at $t + 1$ is $1 - a_i - b_i$, since a tree can only remain in the same class, grow into a higher class, or die. The time unit used is short enough that no tree can skip one diameter class.

Table 8.2 shows specific values of the parameters a_i and b_i that apply to the stand in Table 8.1. The parameters are based on observations from permanent plots that were remeasured several times. In Table 8.2, $a_1 = 0.80$

TABLE 8.2 FRACTION OF TREES STAYING IN THE SAME DIAMETER CLASS, GROWING INTO A THE NEXT DIAMETER CLASS OR DYING WITHIN 5 YEARS

Diameter class, i	Fraction staying, a_i	Fraction growing, b_i	Fraction dying, $1 - a_i - b_i$
1	0.80	0.04	0.16
2	0.90	0.02	0.08
3	0.90	0.00	0.10

and $b_1 = 0.04$ mean that on average 80 percent of the trees in the smallest diameter class will be in the same class 5 years later, while 4 percent of the trees will move to the larger class. The remaining 16 percent are expected to die.

With the parameters in Table 8.2, the growth model (8.1) becomes:

$$\begin{aligned} y_{1,t+1} &= 0.80y_{1,t} + O_t \\ y_{2,t+1} &= 0.04y_{1,t} + 0.90y_{2,t} \\ y_{3,t+1} &= 0.02y_{2,t} + 0.90y_{3,t} \end{aligned} \tag{8.2}$$

To complete the model, we need an expression of the ingrowth, O_t. Biometric studies have shown that ingrowth is very erratic. Nevertheless, other things being equal, ingrowth tends to decrease with the basal area and to increase with the number of trees per unit area. For our sugar maple forest it was found that, on average:

$$\underset{(\text{trees/acre/5 years})}{O_t} = 40 - \underset{(\text{ft}^2/\text{acre})}{0.9B_t} + \underset{(\text{trees/acre})}{0.3N_t}$$

where B_t and N_t are the basal area and number of trees, and the ingrowth is measured over a 5-year period. The relationship shows that, for two stands with the same number of trees per acre, ingrowth is smaller where the trees are larger, and thus the basal area per acre is higher. This is what we expect since in mature stands the thick canopy of the larger trees tends to suppress the growth of younger ones. On the other hand, for two stands of the same basal area per acre, the one with more trees per acre is a younger stand in which ingrowth is more active.

This expression of ingrowth can be changed readily into one involving $y_{1,t}$, $y_{2,t}$, and $y_{3,t}$ only, since:

$$N_t = y_{1,t} + y_{2,t} + y_{3,t}.$$

and

$$B_t = 0.20y_{1,t} + 0.66y_{2,t} + 1.40y_{3,t}$$

where each coefficient is the basal area of the average tree in the corresponding diameter class (see Table 8.1). Thus:

$$O_t = 40 + 0.12y_{1,t} - 0.29y_{2,t} - 0.96y_{3,t}$$

Therefore, the final expression of the growth model is:

$$\begin{aligned} y_{1,t+1} &= 0.92y_{1,t} - 0.29y_{2,t} - 0.96y_{3,t} + 40 \\ y_{2,t+1} &= 0.04y_{1,t} + 0.90y_{2,t} \\ y_{3,t+1} &= 0.02y_{2,t} + 0.90y_{3,t} \end{aligned} \tag{8.3}$$

This basic growth model involves only variables describing the state of the stand at time t and $t + 1$. We shall use it in the next section to describe the growth of an undisturbed stand. Then, we shall use the model to determine the best cutting regime for different management objectives.

8.3 PREDICTING THE GROWTH OF AN UNMANAGED STAND

Stand Dynamics

Let $y_{1,0}$, $y_{2,0}$, and $y_{3,0}$ be the state of an uneven-aged stand at $t = 0$. We would like to predict its future state if it is not cut. To do this we can apply the basic growth relationship (8.3) iteratively. For example, to predict the undisturbed growth of the sugar maple forest displayed in Table 8.1 we set the initial conditions at:

$$y_{1,0} = 420 \qquad y_{2,0} = 117 \qquad y_{3,0} = 7 \text{ (trees per acre)}$$

Replacing these initial conditions in the growth equations gives the number of trees per acre after 5 years:

$$\begin{aligned} y_{1,1} &= 0.92 \times 420 - 0.29 \times 117 - 0.96 \times 7 + 40 = 385.8 \\ y_{2,1} &= 0.04 \times 420 + 0.90 \times 117 = 122.1 \\ y_{3,1} &= 0.02 \times 117 + 0.90 \times 7 = 8.6 \end{aligned}$$

Substituting these values of $y_{1,1}$, $y_{2,1}$, and $y_{3,1}$ in the growth equations would in turn give us the state of the stand after 10 years, $y_{1,2}$, $y_{2,2}$, and $y_{3,2}$. We can proceed in that way for as long as we want. The recursive equations are easy to program in an electronic spreadsheet for microcomputers. Alternatively, one can write a more general simulation program, as discussed in Chap. 13. Regardless of the numerical solution used, the approach is general and can be applied to a model with as many diameter classes as necessary.

Figures 8.1 and 8.2 show predictions of basal area and number of trees per acre, starting with the stand in Table 8.1. That stand was logged heavily in the recent past. As a result, there were initially many trees in the smallest diameter class. If the forest were to grow undisturbed for 50 years, the number of trees in the smallest class would decline considerably. The number in the middle class would remain about constant, and that in the largest class would increase. The data for basal area, in Fig. 8.2, show the increasing importance of the largest diameter class in terms of occupation of the site. It is this dominance of the two largest diameter classes that leads to a subsequent decline in ingrowth and thus to a decline in the number of trees in the smallest diameter class.

Steady State

Figures 8.1 and 8.2 show the data for only 50 years. Pursuing the calculations much longer shows that the number of trees and basal areas oscillate with very

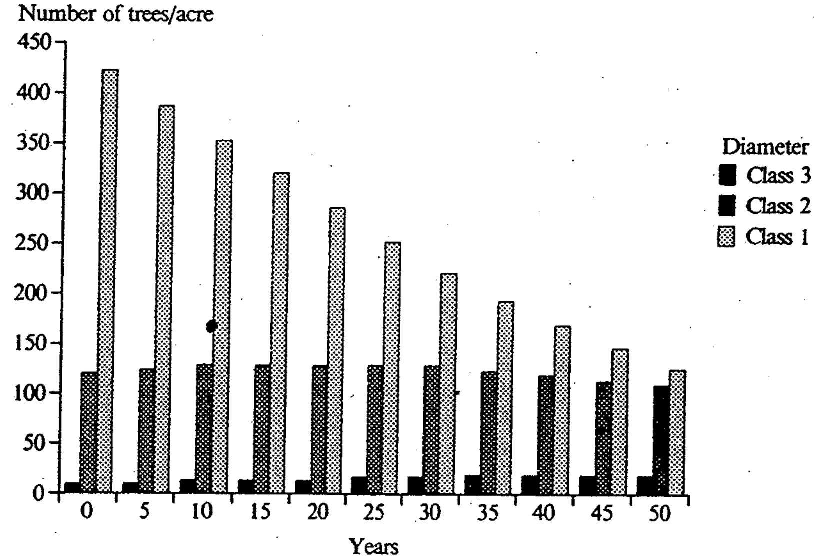

Figure 8.1 Changes in trees per acre in an unmanaged stand.

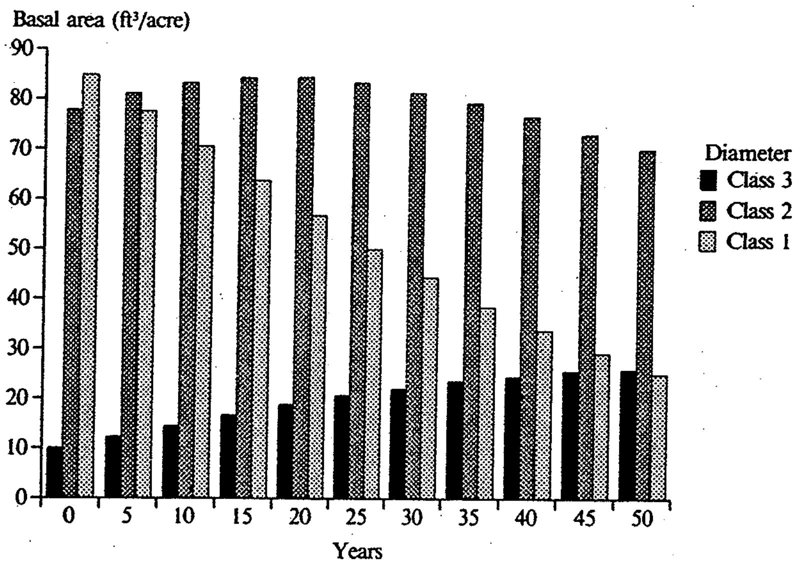

Figure 8.2 Changes in basal area per açre in an unmanaged stand.

long periods but decrease in amplitude, converging toward a steady state in which the stand remains unchanged forever.

There is a more direct way to determine the steady-state forest. By definition, *steady state* means that, regardless of the time when the stand is observed, it has always the same number of trees in each diameter class. That is:

$$y_{i,t+1} = y_{i,t} = y_i \qquad \text{for } i = 1, 2, 3, \text{and for all } t$$

Substituting $y_{i,t+1}$ and $y_{i,t}$ by their unknown, steady-state value, y_i, in the growth model gives:

$$\begin{aligned} y_1 &= 0.92y_1 - 0.29y_2 - 0.96y_3 + 40 \\ y_2 &= 0.04y_1 + 0.90y_2 \\ y_3 &= \qquad\qquad 0.02y_2 + 0.90y_3 \end{aligned}$$

This is a system of three equations in three unknowns that can be solved by substitution. The third equation yields $y_2 = 5y_3$; the second $y_1 = 2.5y_2$; which implies $y_1 = 12.5y_3$. This shows that, in the steady-state forest, there are 12.5 times as many trees in the smallest diameter class as in the largest, and 2.5 times as many as in the intermediate.

Replacing y_1 and y_2 in the first equation by their expression in terms of y_3 gives $y_3 = 11.7$, which in turn implies $y_2 = 58.5$ and $y_1 = 146.2$ trees per acre.

The steady-state distribution has the classical inverse J shape of uneven-aged stands. However, compared with the initial forest, it has fewer trees in the two smallest diameter classes and more in the largest. This is plausible, since the stand we started with is currently being managed and has its largest trees removed periodically.

8.4 GROWTH MODEL FOR A MANAGED STAND

We shall now adapt the model to predict the growth of a stand that is cut periodically. This will be done in two steps, first establishing the relationships that govern the growth of a managed uneven-aged stand and then determining the equations that define the steady state for such a stand.

Growth Equations

The harvest at a certain point in time t is described by the number of trees cut in each diameter class. In our example, this is $h_{1,t}$, $h_{2,t}$ and $h_{3,t}$, where $h_{i,t}$ is the number of trees cut from diameter class i at time t. The number of trees left after the cut in each diameter class i is thus $y_{i,t} - h_{i,t}$. These remaining trees develop according to the growth equations (8.3). Consequently, the growth of an uneven-aged stand that is cut periodically can be described by the following system of equations:

$$\begin{aligned} y_{1,t+1} &= 0.92(y_{1,t} - h_{1,t}) - 0.29(y_{2,t} - h_{2,t}) - 0.96(y_{3,t} - h_{3,t}) + 40 \\ y_{2,t+1} &= 0.04(y_{1,t} - h_{1,t}) + 0.90(y_{2,t} - h_{2,t}) \\ y_{3,t+1} &= \qquad\qquad\qquad\qquad 0.02(y_{2,t} - h_{2,t}) + 0.90(y_{3,t} - h_{3,t}) \end{aligned} \tag{8.4}$$

This system of recursive equations describes the evolution of the stand under any sequence of harvests, regardless of their timing and level, as long as $h_{i,t} \le y_{i,t}$. However, in this chapter we shall concentrate on harvest sequences that maintain the forest in a steady state (the classical sustained-yield management) and on cuts that occur at regular intervals. We shall use first a cutting cycle of 5 years, the time unit of the growth model, and show later how this can be changed.

Steady State for a Managed Stand

A managed uneven-aged stand is in a steady state if the amount cut from it is just equal to the amount by which the stand has grown since the last time that it was cut (Fig. 8.3).

This must be true for each diameter class. Let y_i be the number of trees in diameter class i before harvest and h_i be the number of trees cut in the steady state, then for any t:

$$y_{i,t+1} = y_{i,t} = y_i \qquad \text{and} \qquad h_{i,t+1} = h_{i,t} = h_i \qquad \text{for } i = 1,2,3$$

which, substituted in the growth model (8.4), gives:

$$\begin{aligned} y_1 &= 0.92(y_1 - h_1) - 0.29(y_2 - h_2) - 0.96(y_3 - h_3) + 40 \\ y_2 &= 0.04(y_1 - h_1) + 0.90(y_2 - h_2) \\ y_3 &= \qquad\qquad\qquad 0.02(y_2 - h_2) + 0.90(y_3 - h_3) \end{aligned} \tag{8.5}$$

This is a system of three equations with six unknowns. The only meaningful solutions are those such that:

$$h_i \leq y_i \qquad \text{for } i = 1,2,3 \tag{8.6}$$

because the number of trees cut from each class cannot exceed what is available. There is more than one solution to equations (8.5) and inequalities (8.6), that is, more than one combination of growing stock (y_1, y_2, y_3) and harvest (h_1, h_2, h_3) that maintain a steady state. Our goal is to find the solution that best meets specific objectives.

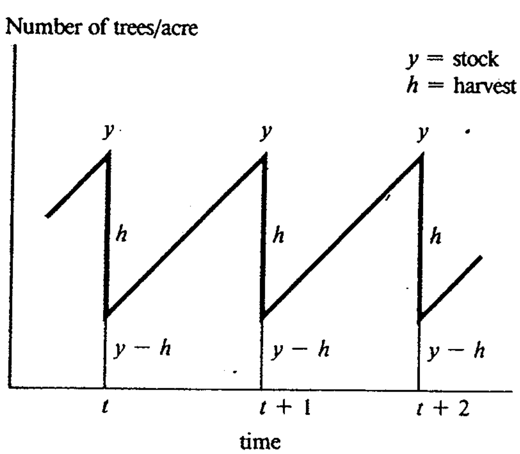

Figure 8.3 Steady state in a managed uneven-aged stand.

8.5 MAXIMIZING THE VOLUME PRODUCED

A classical goal of sustained-yield management is to maximize the volume produced per unit of time while maintaining the forest in a steady state. In our example this means that the stand is restored every 5 years to the state it was in 5 years earlier, and the volume cut every 5 years is constant. Table 8.3 shows the volume, in board feet (bft) of the average tree in each diameter class. The total volume cut from the forest every 5 years is:

$$\underset{(\text{bft})}{Z_Q} = \underset{(\text{bft/tree})}{20} \underset{(\text{trees})}{h_1} + 100h_2 + 300h_3$$

The object is to find the harvest h_1, h_2, h_3 that maximizes this function while satisfying the equations (8.5) and inequalities (8.6). Rearranging the variables in the usual linear-programming format gives the linear-programming tableau in Table 8.4. The optimum solution of this linear program is:

Growing trees per acre: $y_1^* = 500,\ y_2^* = 20,\ y_3^* = 0$
Trees cut per acre per 5 years: $h_1^* = 0,\ h_2^* = 20,\ h_3^* = 0$
Volume produced per acre per 5 years: $Z_Q^* = 2000$ bft

Thus, the optimum stand structure consists in having, just before harvest, 500 trees per acre in the smallest diameter class, 20 in the medium, and nothing in the largest. The optimum cutting rule is to remove every 5 years all the trees from the medium diameter class. This leads to a maximum constant production of 400 board feet per acre per year.

TABLE 8.3 VOLUME AND VALUE OF AVERAGE TREES BY DIAMETER CLASS

Diameter class, i	Volume, q_i (bft)	Value, v_i ($)
1	20	0.15
2	100	4.00
3	300	10.00

TABLE 8.4 LINEAR-PROGRAMMING TABLEAU TO MAXIMIZE VOLUME WITH A CUTTING CYCLE OF 5 YEARS

	y_1	y_2	y_3	h_1	h_2	h_3	
Cla1	0.08	0.29	0.96	0.92	−0.29	−0.96	= 40
Cla2	−0.04	0.10		0.04	0.90		= 0
Cla3		−0.02	0.10		0.02	0.90	= 0
Gec1	1			−1			≥ 0
Gec2		1			−1		≥ 0
Gec3			1			−1	≥ 0
Z_Q				20	100	300	

Using the growth model, verify that in 5 years, the stand is restored to its initial structure just before the harvest.

8.6 ECONOMIC HARVEST AND CUTTING CYCLE

The object here is to determine the structure of the steady-state forest and the corresponding harvest that maximizes the net present value of returns to the owner. We shall proceed in two steps: (1) finding the best stock and harvest for a given cutting cycle, and then (2) determining the effect of changing the cutting cycle.

Maximizing Present Value for a Given Cutting Cycle

For the sugar maple stand in our example, using the data in Table 8.3, the value of the harvest cut periodically in a steady state is:

$$V_H = 0.15h_1 + 4h_2 + 10h_3 \; (\$/a)$$

This value recurs every 5 years and can be sustained forever. Thus, the present value of all the harvests starting at time zero is, assuming an interest rate of 5 percent per year:

$$P_H = V_H + \frac{V_H}{1.05^5 - 1} = V_H \frac{1.05^5}{1.05^5 - 1},$$

which leads to:

$$P_H = 0.69h_1 + 18.5h_2 + 46.2h_3 \; (\$/a)$$

However, to receive this the owner must invest the initial growing stock, which has a value of:

$$V_S = 0.15y_1 + 4y_2 + 10y_3 \; (\$/a)$$

This is undoubtedly a cost since the money tied up in growing stock could be used in other ways. Consequently, what the owner needs to maximize is the present value of the harvests, net of the investment in growing stock, that is:

$$Z_{PV} = P_H - V_S$$

$$\underset{(\$/a)}{Z_{PV}} = 0.69h_1 + 18.5h_2 + 46.2h_3 - 0.15y_1 - 4y_2 - 10y_3 \qquad (8.7)$$

This is the new objective function. The constraints on the possible values of y_1, y_2, y_3 and h_1, h_2, h_3 remain the same as in Table 8.4; they ensure a sustained yield. The optimum solution for this program is:

Growing trees per acre: $y_1^* = 500,\ y_2^* = 20,\ y_3^* = 0$

Trees cut per acre per 5 years: $h_1^* = 0,\ h_2^* = 20,\ h_3^* = 0$

Present value: $Z_{PV}^* = 215 \; (\$/a)$